

# BOUNDS ON HARBOURNE INDICES

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**ABSTRACT.** We continue our study of the Bounded Negativity Conjecture from the point of view of Harbourne constants. We focus our attention on configurations of smooth curves having transversal intersection points only. Generalizing previous work of the authors [5], we give a bound on the so-called Harbourne indices of such configurations on any surface of non-negative Kodaira dimension. We also study in detail configurations of lines on smooth surfaces in  $\mathbb{P}_{\mathbb{C}}^3$ , and we provide a sharp and uniform bound on the Harbourne constants, which only depends on the degree of the surface. Finally, we move on to considering configurations of pseudolines in the real projective plane, and we deduce a uniform bound for their Harbourne constants.

## 1. INTRODUCTION

In this article, we carry on with our study of the local negativity phenomenon for algebraic surfaces. This is strictly related to the following celebrated conjecture, which dates back to the beginning of XX century.

**Conjecture 1.1** (Bounded Negativity Conjecture). *Let  $X$  be a smooth projective surface defined over a field of characteristic zero. Then there exists an integer  $b(X) \in \mathbb{Z}$  such that for all reduced curves  $C \subset X$  one has  $C^2 \geq -b(X)$ .*

This conjecture is known to hold for some class of surfaces, for instance surfaces with  $\mathbb{Q}$ -effective anti-canonical divisor. However, even in those cases, the conjecture is widely open if we start blowing-up points. As an example, if we consider the blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  at  $\geq 10$  general points, then it is not known whether it has Bounded Negativity (BN).

In [2], the authors have introduced the notion of Harbourne constants<sup>1</sup>, which allows one to (potentially) study the property (BN) for all blow-ups of a given surface simultaneously. They should also be thought of as an asymptotic version of the self-intersection numbers, and they provide a more effective approach to the study of the BNC.

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<sup>1</sup>In fact, they defined the so-called Hadean constants, although this notion should be attributed to B. Harbourne.

**Definition 1.2.** Let  $X$  be a smooth projective surface, and consider a non-empty collection  $\mathcal{P} = \{P_1, \dots, P_s\}$  of mutually distinct points on  $X$ . Then, the *Harbourne constant at  $\mathcal{P}$*  is defined to be

$$H(X; \mathcal{P}) := \inf_C \frac{\tilde{C}^2}{s},$$

where  $\tilde{C}$  is the strict transform of  $C$  in the blow-up of  $X$  at the points of  $\mathcal{P}$ , and the infimum is taken over all reduced (possibly reducible) curves on  $X$ . The *global Harbourne constant of  $X$*  is the quantity

$$H(X) := \inf_{\mathcal{P}} H(X; \mathcal{P}),$$

where the infimum is taken over all non-empty collections of mutually distinct points on  $X$ .

The importance of the notion of Harbourne constants in the study of the BNC is highlighted in the following remark: if  $H(X) > -\infty$ , then, for any collection  $\mathcal{P}$  of mutually distinct points on  $X$ , the BNC holds on the blow-up of  $X$  at the points of  $\mathcal{P}$ . Nevertheless, even if  $H(X) = -\infty$ , BNC might still be true on  $X$  or any of its blow-ups at mutually distinct points.

In this note, we propose the notion of Harbourne index attached to a configuration of smooth curves having transversal intersection points. This is a quite natural variation of the original Harbourne constants.

**Definition 1.3.** Let  $X$  be a smooth projective surface. Let  $\mathcal{C} = \{C_1, \dots, C_k\}$  be a transversal configuration of smooth curves on  $X$  and let  $C = \sum_{i=1}^k C_i$  be its associated divisor. The rational number

$$h(X; \mathcal{C}) = h(\mathcal{C}) = \frac{C^2 - \sum_{P \in \text{Sing}(C)} (\text{mult}_P(C))^2}{s} \quad (1)$$

is the *Harbourne index of the transversal configuration  $\mathcal{C} \subset X$* .

We prove three results providing explicit bounds on Harbourne indices, especially we show that, for connected configurations of lines on smooth surfaces of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^3$ , Harbourne indices are uniformly bounded by  $-d(d-1)$ , which is somehow surprising. We also analyze the case of non-connected configurations of lines: we are able to prove that, for an arbitrary configuration of lines, the uniform bound on the Harbourne indices is  $-2d^3 + 7d^2 - 6d - 2$ . The second main result of this note gives a bound on Harbourne indices for configurations of smooth curves having only transversal intersection points on projective surfaces with non-negative Kodaira dimension. In [10] the authors have shown a certain result in a similar flavor to ours. However, they use different tools and methods comparing with our approach since the main difference is made by the so-called logarithmic Miyaoka-Yau inequality. Since settings for these results are different (i.e. we consider a different class of curves), our results are to be considered independently. The last result gives us a uniform bound on Harbourne indices for the so-called pseudoline configurations on the real projective plane.

## 2. HARBOURNE INDICES FOR SURFACES WITH NON-NEGATIVE KODAIRA DIMENSION

In a previous paper [5], the authors have studied the local negativity for configurations of smooth rational curves on K3 and Enriques surfaces. The following result is a generalization of [5, Theorem 2.2], whose proof is based on an idea of Miyaoka [7, Section 2.4].

**Theorem 2.1.** *Let  $X$  be a smooth complex projective surface of non-negative Kodaira dimension, and let  $\mathcal{C} \subset X$  be a configuration of smooth curves of genus  $g$  having  $n$  irreducible components  $C_1, \dots, C_n$  and only transversal intersection points. Then, setting  $C := C_1 + \dots + C_n$ , we have*

$$K_X^2 + K_X \cdot C + 4n(1 - g) - t_2 + \sum_{r \geq 3} (r - 4)t_r \leq 3c_2(X).$$

*Proof.* Let  $\mathcal{C} = \{C_1, \dots, C_n\}$  be a configuration of smooth curves on  $X$  such that each irreducible component has genus  $g$ . If  $\text{Sing}(\mathcal{C})$  denotes the set of singular points of the configuration, we define  $S = \{p_j\}_{j=1}^k$  to be the subset of points in  $\text{Sing}(\mathcal{C})$  with multiplicity  $\geq 3$ . Consider the blow-up of  $X$  at  $S$ , namely

$$\sigma : Y \longrightarrow X;$$

under the pull-back along  $\sigma$ , the configuration  $\mathcal{C}$  on  $X$  yields a configuration  $\sigma^*\mathcal{C}$  which consists of the strict transforms of the  $C_i$ 's and the exceptional divisors. Notice that  $\sigma^*\mathcal{C}$  is again a configuration of smooth curves that admits only double points as singularities. Following [7, Section 2.4], we set  $\tilde{C} := \tilde{C}_1 + \dots + \tilde{C}_n$ . The idea is to use the Miyaoka-Yau inequality

$$3c_2(Y) - 3e(\tilde{C}) \geq (K_Y + \tilde{C})^2,$$

and thus we now need to compute the terms in the above inequality. We see that

$$\begin{aligned} c_2(Y) &= c_2(X) + k, \\ e(\tilde{C}) &= (2 - 2g)n - t_2, \end{aligned}$$

which yield  $c_2(Y) - e(\tilde{C}) = c_2(X) + k - (2 - 2g)n + t_2$ .

Notice that, since  $\kappa(X) \geq 0$ , we have that  $|mK_X| \neq \emptyset$ , and thus  $mK_X$  is linearly equivalent to an effective divisor  $D$ . Therefore,

$$K_Y + \tilde{C} = (\sigma^*K_X + E) + \tilde{C} = \sigma^*K_X + (E + \tilde{C}) = \frac{1}{m}\sigma^*D + (E + \tilde{C}),$$

where  $E := \sum_{j=1}^k E_j$  is the sum of all exceptional divisors. It follows that  $K_Y + \tilde{C}$  is numerically equivalent to a rational effective divisor, which in turn allows us to use the (logarithmic) Miyaoka-Yau inequality according to [7, Corollary 1.2].

We now have:

$$\begin{aligned}
K_Y + \tilde{C} &= \sigma^*(K_X + C) - \sum_{j=1}^k (m_j - 1)E_j, \\
(K_Y + \tilde{C})^2 &= (K_X + C)^2 - \sum_{j=1}^n (m_j - 1)^2 \\
&= K_X^2 + 2K_X.C + \sum_j C_j^2 + 2t_2 + \sum_{j=1}^s (m_j - 1) \\
&= K_X^2 + K_X.C + (2g - 2)n + 2t_2 + \sum_{j=1}^s (m_j - 1).
\end{aligned}$$

By the Miyaoka-Yau inequality, we see that

$$K_X^2 + K_X.C + (2g - 2)n + 2t_2 + \sum_{j=1}^s (m_j - 1) \leq 3(c_2(X) + k - (2 - 2g)n + t_2)$$

and finally

$$K_X^2 + K_X.C + 4n(1 - g) - t_2 + \sum_{r \geq 3} (r - 4)t_r \leq 3c_2(X),$$

which completes the proof.  $\square$

Now we can prove the first main result of this paper.

**Theorem 2.2.** *In the setting of the previous theorem, one has*

$$h(X; \mathcal{C}) \geq -4 + \frac{K_X^2 - 3c_2(X) + 2(1 - g)n + t_2}{s}$$

*Proof.* Let  $\mathcal{C} = \{C_1, \dots, C_n\}$ ,  $L := C_1 + \dots + C_n$ , and  $\tilde{L}$  its strict transform in the blow-up at the  $s$  singular points of the configuration. We observe that

$$\tilde{L}^2/s = \frac{L^2 - \sum_j m_j^2}{s} = \frac{\sum_{j=1}^n C_j^2 + I_d - \sum_{r \geq 2} r^2 t_r}{s},$$

where  $I_d := 2 \sum_{i < j} C_i.C_j$  is the number of incidences of the collection  $\mathcal{C}$  of smooth curves on  $X$  with transversal intersections. Obviously one has

$$I_d - \sum_{r \geq 2} r^2 t_r = - \sum_{r \geq 2} r t_r,$$

and moreover, by arguing in a similar way, we can rephrase the bound in Theorem 2.1 in the following way:

$$- \sum_{r \geq 2} r t_r \geq K_X^2 + K_X.C + 4(1 - g)n + t_2 - 4s - 3c_2(X).$$

This yields

$$h(X; \mathcal{C}) \geq -4 + \frac{K_X^2 - 3c_2(X) + 2n(1 - g) + t_2}{s},$$

and we are done.  $\square$

**Remark 2.3.** If  $A$  is a smooth abelian surface and  $\mathcal{C}$  is a configuration of elliptic curves<sup>2</sup>, then one obtains

$$t_2 + t_3 \geq \sum_{r \geq 5} (r - 4)t_r.$$

Moreover, we have the following bound

$$h(A; \mathcal{C}) \geq -4 + \frac{t_2}{s} \geq -4,$$

and as it is pointed out in [11] this bound is sharp. To this end, consider an abelian surface equipped with the complex multiplication given by  $e^{2\pi i/3}$ . Then the equality is provided by the following configuration

$$\mathcal{E} = \{F_1, F_2, \Delta, \Gamma\},$$

where  $F_1, F_2$  are fibers,  $\Delta$  is the diagonal and  $\Gamma$  is the graph of the complex multiplication.

**Remark 2.4.** For the so-called  $k$ -regular configurations of curves, our bound from Theorem 2.2 is finite and depends only on the genus of the curves. Let  $\mathcal{L}$  be a configuration of curves such that the following conditions are satisfied:

- (1) all intersection points are transversal,
- (2) there exists a positive integer  $k$  such that for any pair  $C_1, C_2 \in \mathcal{L}$  (not necessarily distinct) we have  $C_1.C_2 = k$ .

Then  $\mathcal{L}$  is said to be  $k$ -regular. For such configurations G. Urzúa has shown [13, Remark 7.4] that if  $\mathcal{L} \subset X$  is  $k$ -regular and consists of  $n \geq 3$  curves, then  $s \geq n$ . Now we find a bound on Harbourne indices for such configurations.

First of all,

$$\begin{aligned} h(X; \mathcal{L}) &\geq -4 + \frac{K_X^2 - 3c_2(X) + 2n(1 - g) + t_2}{s} \\ &\geq -4 + \frac{K_X^2 - 3c_2(X) + 2n(1 - g)}{s}. \end{aligned}$$

If  $g = 0, 1$ , there is nothing to prove. Assume that  $g \geq 2$ . Since  $K_X^2 - 3c_2(X) \leq 0$  by the Miyaoka-Yau inequality, than in order to find a lower bound on  $h(X; \mathcal{L})$  we need to find a lower bound on  $s$ . Seeing that  $s \geq n$  and

$$h(X; \mathcal{L}) \geq -4 - 2(g - 1) - (3c_2(X) - K_X^2),$$

which completes the proof.

**Remark 2.5.** In [8], Miyaoka proves the Orbibundle Miyaoka-Yau-Sakai inequality for a pair  $(X, C)$ , where  $X$  is a surface of non-negative Kodaira dimension, and  $C$  is an integral curve. His result relates numerical invariants of  $X$ , such as  $K_X^2$  and  $c_2(X)$ , to those of  $C$ , as the canonical degree  $C.K_X$

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<sup>2</sup>A configuration of elliptic curves on an abelian surface has only transversal intersection points: the universal cover is  $\mathbb{C}^2$ , and an elliptic curve is the image of a line.

and the genus  $g(C)$ . In [8, Remark G], the author observes that his result also holds true for configurations of curves  $C = \sum_i C_i$  if one defines the genus of  $C$  by

$$g(C) - 1 = \sum_i (g(C_i) - 1).$$

Notice that, according to this definition, for a configuration  $C = \sum_{i=1}^n C_i$  of curves of genus  $g$ , one has that  $g(C) - 1 = n(g - 1)$ . It can be checked that the proof of Theorem 2.1 also works for inhomogenous configurations, i.e. for configurations of arbitrary curves  $C = C_1 + \dots + C_n$  having transversal intersection points, where now the  $C_i$ 's are allowed to have different genera. Therefore, for such a configuration, one obtains the following analogs of Theorems 2.1 and 2.2:

$$K_X^2 + K_X \cdot C + 4(1 - g(C)) - t_2 + \sum_{r \geq 3} (r - 4)t_r \leq 3c_2(X);$$

$$h(C) \geq -4 + \frac{K_X^2 - 3c_2(X) + 2(1 - g(C)) + t_2}{s}.$$

It is also possible to weaken the hypothesis of  $C$  being a configuration of smooth curves: indeed, a similar result holds if we allow nodal singularities. We leave the computation to the interested reader.

**Remark 2.6.** We now would like to address the accuracy of our bound in Theorem 2.2. In a previous paper [5], the authors have exhibited a variety of examples where the aforementioned bound is quite accurate. Nevertheless, it has been observed that its accuracy can drastically drop, even in very simple examples, and we now provide an instance of this phenomenon. Let  $S_4 \subset \mathbb{P}_{\mathbb{C}}^3$  be the Schur quartic surface, namely the hypersurface of  $\mathbb{P}_{\mathbb{C}}^3$  given by the following equation:

$$S_4 : \quad x^4 - xy^3 = z^4 - zw^3.$$

Let us consider a configuration  $L$  consisting of four lines meeting at a unique point (see, for example, [5, Example 3.5]). By Theorem 2.2, we conclude that  $h(C) \geq -73$ , while a direct computation shows that  $h(C) = -12$ . This observation motivated us in pursuing a detailed study of configurations of lines on hypersurfaces in  $\mathbb{P}_{\mathbb{C}}^3$ , which finally resulted in Section 3.

**Remark 2.7.** Let  $X$  be a smooth projective surface of non-negative Kodaira dimension. There exists a uniform bound for the H-index of (homogenous) configurations of smooth curves of genera  $g = 0, 1$ . Indeed, this readily follows from Theorem 2.2 and the Miyaoka-Yau inequality  $K_X^2 - 3c_2(X) \leq 0$ :

$$\begin{aligned} h(C) &\geq -4 + \frac{K_X^2 - 3c_2(X) + 2n(1 - g) + t_2}{s} \\ &\geq -4 + \frac{K_X^2 - 3c_2(X)}{s} \geq -4 + K_X^2 - 3c_2(X). \end{aligned}$$

### 3. HARBOURNE INDICES FOR LINE CONFIGURATIONS ON SMOOTH SURFACES IN $\mathbb{P}_{\mathbb{C}}^3$

In [2], the authors have shown that if  $\mathcal{L}$  is a line configuration on the complex projective plane, then  $h(\mathbb{P}_{\mathbb{C}}^2; \text{Sing}(\mathcal{L})) \geq -4$ . In particular, this result proves the existence of a uniform bound on the Harbourne constants of lines configurations on  $\mathbb{P}_{\mathbb{C}}^2$ . In this section, we prove that there exists a uniform bound on Harbourne constants for line configurations on smooth hypersurfaces of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^3$ , for any  $d \geq 4$ . Notice that this is a rather strong statement compared to [9, Main Result]. The proof is divided into two parts: in the first one, we deal with the case of connected configurations of lines, whereas in the second one we handle the general case of arbitrary configurations.

First, we need to define connected configurations of lines. Let  $X$  be a smooth surface of degree  $d \geq 4$ , and let  $\mathcal{L}$  be a configuration of lines on  $X$ . We say that  $\mathcal{L}$  is *connected* if the union of the lines in  $\mathcal{L}$  is connected as a subset of  $\mathbb{P}_{\mathbb{C}}^3$ . Let us observe that if  $\mathcal{L}$  is a connected configuration of  $k \geq 2$  lines on  $X$ , then

$$h(X; \mathcal{L}) = -\frac{(d-2)k + \sum r t_r}{\sum t_r}.$$

We move on to stating the most interesting result of this section.

**Theorem 3.1.** *Let  $X$  be a surface in  $\mathbb{P}_{\mathbb{C}}^3$  of degree  $d \geq 4$ , and let  $\mathcal{L}$  be a connected configuration of  $k \geq 2$  lines on  $X$ . Then,*

$$h(\mathcal{L}) \geq -d(d-1).$$

*Moreover, this bound is sharp, and it is achieved by the configuration consisting of  $d$  lines meeting at a single point.*

In fact, we will prove something stronger, from which Theorem 3.1 will follow as a corollary.

**Theorem 3.2.** *Under the assumptions above, suppose that  $m \in \mathbb{N}$  is the positive integer such that  $t_m \neq 0$  and  $t_r = 0$  for all  $r > m$ . Then,*

$$h(\mathcal{L}) \geq -m(d-1).$$

*Moreover, this bound is sharp, and it is achieved by the configuration consisting of  $m$  lines meeting at a single point.*

*Proof.* Recall that given a line  $\ell \subset X$ ,  $\ell^2 = 2 - d$ . Let  $P$  be an  $m$ -point of  $\mathcal{L}$ . Since  $X$  is smooth, the  $m$  lines meeting at  $P$  must be coplanar. Let us call these lines  $\ell_1, \dots, \ell_m$ , and consider the plane  $\Pi$  containing them. The hyperplane section  $X \cap \Pi$  can be written as

$$X \cap \Pi = \ell_1 \cup \dots \cup \ell_m \cup \ell'_1 \cup \dots \cup \ell'_e \cup C_1 \cup \dots \cup C_s,$$

where  $e \geq 0$ ,  $s \geq 0$  and

$$m + e + \sum_{i=1}^s \deg(C_i) = d.$$

After possibly renumbering, the lines  $\ell'_1, \dots, \ell'_{\bar{e}}$  belong to the configuration  $\mathcal{L}$  ( $0 \leq \bar{e} \leq e$ ). Set

$$\mathcal{M} := \{\ell_1, \dots, \ell_m, \ell'_1, \dots, \ell'_{\bar{e}}\},$$

and  $\mathcal{N} := \mathcal{L} \setminus \mathcal{M}$ . Finally, letting  $k \geq 2$  be the number of lines in  $\mathcal{L}$ , put  $n := k - m - \bar{e}$ . Our aim is to find a non-negative integer  $b$  such that  $h(\mathcal{L}) \geq -b$ . This quantity will depend on the degree  $d$  and the complexity of our configuration, namely the maximum number of lines in  $\mathcal{L}$  intersecting at a single point.

Notice that

$$h(\mathcal{L}) = -\frac{(d-2)k + \sum r t_r}{\sum t_r} = -\frac{(d-2)(n+m+\bar{e}) + \sum r t_r}{\sum t_r},$$

and assume that  $h(\mathcal{L}) \geq -b$ . This condition is equivalent to

$$(d-2)(n+m+\bar{e}) + \sum (r-b)t_r \leq 0. \quad (2)$$

Let  $M$  (respectively  $N$ ) be the divisor associated to the configuration  $\mathcal{M}$  (respectively  $\mathcal{N}$ ). We will now proceed with splitting the summations according to whether a  $r$ -point lies on  $M \setminus N$ ,  $M \cap N$  or  $N \setminus M$ . To this end, for a set  $S$ , let us denote by  $t_{r,S}$  the number of  $r$ -points that lie in  $S$ . Then, one has:

$$\begin{aligned} \sum r t_r &= \sum r t_{r,M \setminus N} + \sum r t_{r,M \cap N} + \sum r t_{r,N \setminus M}; \\ \sum t_r &= \sum t_{r,M \setminus N} + \sum t_{r,M \cap N} + \sum t_{r,N \setminus M}. \end{aligned}$$

Now observe that we can write  $n$  as

$$n = n_R + \sum (r-1)t_{r,M \cap N}, \quad (3)$$

where  $n_R$  is the number of lines in  $\mathcal{N}$  intersecting one of the  $\ell'_i$ 's for  $i = \bar{e} + 1, \dots, e$ , or one of the  $C_i$ 's for  $i = 1, \dots, s$ . This follows because, if  $Q$  is a point where a line in  $\mathcal{M}$  and a line in  $\mathcal{N}$  intersect, then there is no other line in  $\mathcal{M}$  going through  $Q$  (otherwise  $Q$  would be singular). Moreover, we can bound  $n_R$  as

$$n_R \leq \sum r t_{r,N \setminus M}. \quad (4)$$

Indeed, as the configuration is connected, on each line contributing to  $n_R$  there is an  $r$ -point in  $N \setminus M$ , for some  $r \geq 2$ , and taking the weighted sum yields the estimate.

Now, notice that the quantities  $t_{r,M \setminus N}$  only depend on the configuration  $M$ , which is a configuration of lines in  $\mathbb{P}_{\mathbb{C}}^2$ . We aim at giving an upper bound for the number of lines of the configuration ( $m + \bar{e}$  in our case) in terms of the  $t_r$ 's. In general, letting  $\mathcal{C}$  be a configuration of  $c \geq 2$  lines in  $\mathbb{P}_{\mathbb{C}}^2$ , and  $\ell$  a line in  $\mathcal{C}$ , one has:

$$c = 1 + \sum (r-1)t_{r,\ell}. \quad (5)$$



Moreover, unless  $\mathcal{C}$  is the configuration of  $c$  lines meeting at one point,

$$c = 1 + \sum (r-1)t_{r,\ell} \leq \sum r t_r, \quad (6)$$

as there exists at least an  $r$ -point outside of  $\ell$ , for some  $r \geq 2$ . Nevertheless, the same estimate holds even in the case of all lines meeting at a single point. Then, we can apply (3), (4) and (6) to the left-hand side of (2) to get:

$$\begin{aligned} & (d-2)(n+m+\bar{e}) + \sum (r-b)t_r \\ & \leq \sum [(d-1)r - (d-2+b)]t_{r,M \cap N} \\ & \quad + \sum [(d-1)r - b](t_{r,M \setminus N} + t_{r,N \setminus M}). \end{aligned}$$

This shows that it is enough to set  $b := m(d-1)$  to satisfy condition (2), and this completes the proof of the existence of the bound.

We are left to show that this bound is actually sharp. The condition  $h(\mathcal{L}) = -m(d-1)$  is equivalent to

$$\begin{aligned} & \sum [(d-1)(r-m-1) + 1]t_{r,M \cap N} \\ & + \sum [(d-1)(r-m)](t_{r,M \setminus N} + t_{r,N \setminus M}) = 0 \end{aligned}$$

As the first summation is always negative and the second is non-positive, it follows that  $M \cap N = \emptyset$ . By the connectedness of  $\mathcal{L}$ , this in turn implies that  $N = \emptyset$ , and thus

$$\sum [(d-1)(r-m)]t_{r,M} = 0.$$

This equality implies that  $\mathcal{L} = \mathcal{M}$  must be the configuration of  $m$  lines meeting at a single point.  $\square$

Let us present the following example, which shows that the bound in Theorem 3.1 is sharp.

**Example 3.3.** (Schur degree- $d$  surface) For a positive integer  $d$  let us consider the following surface

$$S_d: x^d - xy^{d-1} = z^d - zw^{d-1}.$$

We will call this surfaces as the Schur degree- $d$  surface. Observe that for  $d = 4$  we recover the celebrated Schur quartic surface. By intersecting  $X$  with the hyperplane  $\{x = 0\}$  one obtains a configuration  $\mathcal{L}$  of  $d$  lines  $\ell_1, \dots, \ell_d$  intersecting at one  $d$ -fold point. By the adjunction formula we can compute that each line has self-intersection  $\ell_i^2 = 2 - d$ . Therefore,

$$h(S_d; \mathcal{C}_d) = d - d^2.$$

Also notice that, as the degree becomes larger, the Harbourne index gets more negative, i.e.

$$\lim_{d \rightarrow +\infty} h(S_d; \mathcal{C}_d) = -\infty.$$

This phenomenon was already observed in [9, Example 3.4]. However, in that example one has that the Harbourne index decreases like  $-d/2$  as  $d \rightarrow +\infty$ , whereas for the degree  $d$ -Schur surface we have

$$h(S_d; \mathcal{C}_d) \sim -d^2$$

as  $d \rightarrow +\infty$ , hence quadratic growth for the negativity.

So far, we have dealt with connected configurations of lines only. We would like to obtain a bound which holds for an arbitrary configuration of lines, possibly non-connected. The first step toward such a result is showing that the bound  $h(X; \mathcal{L}) \geq -d(d-1)$  holds for a configuration  $\mathcal{L}$  whose connected components consist of at least two lines.

**Lemma 3.4.** *Let  $X$  be a surface of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^3$ , and let  $\mathcal{L} = \mathcal{L}_1 \sqcup \dots \sqcup \mathcal{L}_n$  ( $n \geq 1$ ) be a configuration of lines whose connected components are the  $\mathcal{L}_i$ 's and such that each  $\mathcal{L}_i$  consists of at least two lines. Then,*

$$h(X; \mathcal{L}) \geq -d(d-1).$$

*Proof.* Set  $s_i := \#\text{Sing}(\mathcal{L}_i)$  ( $1 \leq i \leq n$ ), and  $s := \sum_{i=1}^n s_i$ . Let  $L$  be the divisor associated to  $\mathcal{L}$ , and let  $L_i$  be the one associated to  $\mathcal{L}_i$  ( $i = 1, \dots, n$ ). The existence of at least two lines in each  $\mathcal{L}_i$  guarantees that  $s_i > 0$  ( $1 \leq i \leq n$ ). Then, by means of Theorem 3.1, we get

$$\begin{aligned} h(\mathcal{L}) &= \frac{\sum_{C \in \mathcal{L}} C^2 - \sum rt_{r,L}}{s} \\ &= \sum_{i=1}^n \frac{\sum_{C \in \mathcal{L}_i} C^2 - \sum rt_{r,L_i}}{s} \\ &= \frac{1}{s} \sum_{i=1}^n s_i \frac{\sum_{C \in \mathcal{L}_i} C^2 - \sum rt_{r,L_i}}{s_i} \\ &\geq -\frac{1}{s} \sum_{i=1}^n d(d-1)s_i = -d(d-1). \end{aligned}$$

□

We are left to handle the case in which some of the  $\mathcal{L}_i$ 's consist of an isolated line (and thus  $s_i = 0$ ). These isolated lines give non-trivial contribution to the Harbourne index.

**Theorem 3.5.** *Let  $X$  be a surface of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^3$ , and let*

$$\mathcal{L} = \mathcal{L}_1 \sqcup \dots \sqcup \mathcal{L}_m \sqcup \mathcal{L}_{m+1} \sqcup \dots \sqcup \mathcal{L}_{m+n} \quad (m \geq 1, n \geq 0)$$

*be a configuration of lines whose connected components are the  $\mathcal{L}_i$ 's, and such that  $\mathcal{L}_i$  consists of at least two lines for  $i = 1, \dots, m$ , and  $\mathcal{L}_i$  is an isolated line for  $i = m+1, \dots, m+n$ . Then,*

$$h(X; \mathcal{L}) \geq -d(d-1) + (2-d)n.$$

*Proof.* The assumptions imply that  $s_i > 0$  for  $i = 1, \dots, m$ , and  $s_i = 0$  otherwise. By means of Lemma 3.4, we have

$$\begin{aligned}
h(\mathcal{L}) &= \frac{\sum_{\mathcal{L}} C^2 - \sum_{\mathcal{L}} r t_r}{s_1 + \dots + s_m} \\
&= \sum_{i=1}^m \frac{\sum_{\mathcal{L}_i} C^2 - \sum_{\mathcal{L}_i} r t_r}{s_1 + \dots + s_m} + \sum_{i=m+1}^{m+n} \frac{\sum_{\mathcal{L}_i} C^2 - \sum_{\mathcal{L}_i} r t_r}{s_1 + \dots + s_m} \\
&\geq -d(d-1) + \sum_{i=m+1}^{m+n} \frac{\sum_{\mathcal{L}_i} C^2 - \sum_{\mathcal{L}_i} r t_r}{s_1 + \dots + s_m} \\
&= -d(d-1) + \sum_{i=m+1}^{m+n} \frac{2-d}{s_1 + \dots + s_m} \geq -d(d-1) + (2-d)n.
\end{aligned}$$

□

Finally, we are able to deduce a uniform bound for the Harbourne index of line configurations, recovering the same situation as for lines in  $\mathbb{P}_{\mathbb{C}}^2$ .

**Corollary 3.6.** *Let  $X$  be a smooth surface of degree  $d$  in  $\mathbb{P}^3$ , and let  $\mathcal{L}$  be a configuration of lines on  $X$ . Then,*

$$h(X; \mathcal{L}) \geq -2d^3 + 7d^2 - 6d - 2.$$

*Proof.* A result of Miyaoka [7] states that the maximum number of disjoint lines on a hypersurface in  $\mathbb{P}^3$  of degree  $d$  is  $2d(d-2)$ . Let

$$\mathcal{L} = \mathcal{L}_1 \sqcup \dots \sqcup \mathcal{L}_m \sqcup \mathcal{L}_{m+1} \sqcup \dots \sqcup \mathcal{L}_{m+n} \quad (m \geq 1, n \geq 0)$$

be an arbitrary configuration of lines. As  $m \geq 1$ ,  $n \leq 2d(d-2) - 1$ : if  $n = 2d(d-2)$ , as  $m \geq 1$ , there would be  $2d(d-2) + 1$  disjoint lines, which is clearly impossible. Then, by means of Theorem 3.5, we have

$$h(\mathcal{L}) \geq -d(d-1) + (2-d)n \geq -2d^3 + 7d^2 - 6d - 2.$$

□

#### 4. HARBOURNE INDICES FOR CONFIGURATIONS OF PSEUDOLINES IN THE REAL PROJECTIVE PLANE

In the last section we would like to show a certain variation on [2, Theorem 3.15], which gives us a uniform bound on Harbourne constants for line configurations in  $\mathbb{P}_{\mathbb{R}}^2$ .

**Theorem 4.1.** *Let  $\mathcal{L}$  be a configuration of lines in  $\mathbb{P}_{\mathbb{R}}^2$ . Then*

$$h(\mathbb{P}_{\mathbb{R}}^2; \text{Sing}(\mathcal{L})) \geq -3,$$

*and this bound is asymptotically sharp.*

Now we would like to focus on configurations of pseudolines. This class of configurations plays an important role in combinatorics and in the real projective geometry.

**Definition 4.2.** A collection  $\mathcal{C}$  of smooth closed curves on the real projective plane is called a configuration of pseudolines if

- (1) there is no point where all pseudolines intersect,
- (2) the intersection number of two different irreducible components of  $\mathcal{C}$  is equal to 1,
- (3) curves do not have self-intersections.

In a very recent paper Shnurnikov [12] has shown the following result, which can be viewed as a Hirzebruch-type inequality.

**Theorem 4.3.** *Let  $\mathcal{C} \subset \mathbb{P}_{\mathbb{R}}^2$  be a configuration of  $k \geq 3$  pseudolines. Then one has*

$$t_2 + \frac{3}{2}t_3 \geq 8 + \sum_{r \geq 4} (2r - 7.5)t_r.$$

Now we can prove the main result of this section.

**Theorem 4.4.** *Let  $\mathcal{C} \subset \mathbb{P}_{\mathbb{R}}^2$  be a configuration of  $k \geq 3$  pseudolines. Then one has*

$$h(\mathbb{P}_{\mathbb{R}}^2; \text{Sing}(\mathcal{C})) \geq -3.725.$$

*Proof.* We start with computing  $\mathcal{C}^2$ . By [12, Section 2.2] we know that for configurations of pseudolines one has

$$k(k-1) = \sum_{r \geq 2} t_r(r^2 - r).$$

This implies that

$$\begin{aligned} \mathcal{C}^2 &= \sum_{i=1}^k C_i^2 + 2 \sum_{i < j} C_i \cdot C_j = \sum_{i=1}^k C_i^2 + k(k-1) = \sum_{i=1}^k C_i^2 + \sum_{r \geq 2} t_r(r^2 - r) \\ &= \sum_{i=1}^k C_i^2 + f_2 - f_1. \end{aligned}$$

Let us denote for simplicity  $d := \sum_{i=1}^k C_i^2$ . Observe that

$$h(\mathbb{P}_{\mathbb{R}}^2; \text{Sing}(\mathcal{C})) = \frac{\mathcal{C}^2 - f_2}{f_0} = \frac{d - f_2 + f_2 - f_1}{f_0} = \frac{d - f_1}{f_0}.$$

By Shnurnikov's Inequality we have

$$-f_1 \geq 4 + \frac{5}{4}t_2 - 3.725 \cdot f_0,$$

and then

$$h(\mathbb{P}_{\mathbb{R}}^2; \text{Sing}(\mathcal{C})) \geq -3.725 + \frac{d + 4 + \frac{5}{4}t_2}{f_0},$$

which completes the proof.  $\square$

Our result here is even more stronger. We show, similarly to the result for line configurations in [2], that  $-3.725$  provides a *global* bound. Let us recall the following notion.

**Definition 4.5.** Let  $\mathcal{P} = \{P_1, \dots, P_s\} \subset \mathbb{P}_{\mathbb{R}}^2$  be the set of mutually distinct  $s \geq 1$  points and let  $\mathcal{L}$  be a configuration of pseudolines. Then the global Harbourne constant is defined as

$$H_L(\mathbb{P}_{\mathbb{R}}^2) := \inf_{\mathcal{L}} \inf_{\mathcal{P}} \frac{\tilde{\mathcal{L}}^2}{s},$$

where  $\tilde{\mathcal{L}}$  denotes the strict transform of  $\mathcal{L}$  under the blowing up  $f : X \rightarrow \mathbb{P}_{\mathbb{R}}^2$  along  $\mathcal{P}$  and the first infimum is taken over all possible configurations of pseudolines and the second infimum is taken over all possible configurations of points.

**Theorem 4.6.** *Under the notions as above, one has*

$$H_L(\mathbb{P}_{\mathbb{R}}^2) \geq -3.725.$$

Since the proof goes along the same lines as in [2, Theorem 3.3], thus we present a sketch of the argument. The first step is done by the following proposition, which is easy to show.

For  $\mathcal{L} \subset \mathbb{P}_{\mathbb{R}}^2$  a configuration of pseudolines we denote by  $\text{Sing}(\mathcal{L})$  the set of all singular points and by  $\text{Smooth}(\mathcal{L})$  the set of all smooth points.

**Proposition 4.7.** *Let  $\mathcal{L} \subset \mathbb{P}_{\mathbb{R}}^2$  be a configuration of pseudolines. We define  $\mathcal{P} = \text{Sing}(\mathcal{L}) \cup \text{Smooth}(\mathcal{L})$ . Denote by  $f : X \rightarrow \mathbb{P}_{\mathbb{R}}^2$  the blowing-up along  $\mathcal{P}$  and assume that  $\frac{(f^*\mathcal{L})^2 - \sum_{p \in \mathcal{P}} \text{mult}_p(\mathcal{L})^2}{\#\mathcal{P}} \leq -1$ . Then*

$$\frac{(f^*\mathcal{L})^2 - \sum_{p \in \mathcal{P}} \text{mult}_p(\mathcal{L})^2}{\#\mathcal{P}} \geq \frac{(f^*\mathcal{L})^2 - \sum_{p \in \text{Sing}(\mathcal{L})} \text{mult}_p(\mathcal{L})^2}{\#\text{Sing}(\mathcal{L})}$$

The following example shows that our assumption  $\frac{(f^*\mathcal{L})^2 - \sum_{p \in \mathcal{P}} \text{mult}_p(\mathcal{L})^2}{\#\mathcal{P}} \leq -1$  is crucial.

**Example 4.8.** Consider a pencil  $\mathcal{L}_d$  of  $d$ -lines. Then  $\text{Sing}(\mathcal{L}_d)$  consists of a single  $d$ -fold point. Let us define  $\text{Smooth}(\mathcal{L}_d)$  as a set of  $s \geq 1$  smooth points of  $\mathcal{L}_d$ . Denote by  $f : X \rightarrow \mathbb{P}_{\mathbb{R}}^2$  the blowing-up along  $\mathcal{P}$ . It is easy to see that

$$\frac{(f^*\mathcal{L}_d)^2 - \sum_{p \in \text{Sing}(\mathcal{L}_d)} \text{mult}_p(\mathcal{L}_d)^2}{1} = 0.$$

However,

$$\frac{(f^*\mathcal{L}_d)^2 - \sum_{p \in \mathcal{P}} \text{mult}_p(\mathcal{L}_d)^2}{s+1} = \frac{-s}{s+1},$$

which tends to  $-1$  provided that  $s \rightarrow \infty$ .

The above proposition tells us that in order to minimize the global linear Harbourne constant one needs to consider the case when  $\mathcal{P} \subset \text{Sing}(\mathcal{L})$ . We show that actually one needs to consider  $\text{Sing}(\mathcal{L}) = \mathcal{P}$ . To this end, it is enough to observe that if  $\mathcal{P} \subset \text{Sing}(\mathcal{L})$  is a proper subset and  $p' \in \text{Sing}(\mathcal{L}) \setminus \mathcal{P}$ , then the Harbourne constant for  $\mathcal{P} \cup \{p'\}$  is weighted average of the self-intersection number for  $\mathcal{P}$  (which is greater than  $-3.725$  by the previous

considerations) and number which is less equal then  $-4$  since we add at least double point. This completes the proof.

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## REFERENCES

- [1] Th. Bauer & B. Harbourne & A. L. Knutsen & A. Küronya & S. Müller-Stach & X. Roulleau & T. Szemberg: Negative curves on algebraic surfaces. *Duke Math. J.* **162**: 1877 – 1894 (2013).
- [2] Th. Bauer & S. Di Rocco & B. Harbourne & J. Huizenga & A. Lundman & P. Pokora & T. Szemberg: Bounded Negativity and Arrangements of Lines. *International Mathematical Research Notices* **vol. 2015**: 9456 - 9471 (2015), doi:10.1093/imrn/RNU236.
- [3] B. Harbourne: Global aspects of the geometry of surfaces. *Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica* **vol. IX**: 5 – 41 (2010).
- [4] F. Hirzebruch: Arrangement of lines and Algebraic surfaces. Arithmetic and geometry, Vol. II, 113–140, Progr. Math., 36, Birkhäuser, Boston, Mass., 1983.
- [5] R. Laface & P. Pokora, On the local negativity of Enriques and K3 surfaces. **arXiv:1512.06022**.
- [6] Y. Miyaoka: On the Chern numbers of surfaces of general type. *Invent. Math.* **42(1)**: 225 – 237 (1977).
- [7] Y. Miyaoka: The maximal number of quotient singularities on surfaces with given numerical invariants. *Math. Ann.* **268**: 159 – 171 (1984).
- [8] Y. Miyaoka: The Orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan Theorem. *Publ. RIMS, Kyoto Univ.* **44**: 403–417 (2008).
- [9] P. Pokora: Harbourne constants and arrangements of lines on smooth hypersurfaces in  $\mathbb{P}_{\mathbb{C}}^3$ . *Taiwanese J. Math.* **vol. 20(1)**: 25 – 31 (2016), doi:10.11650/tjm.19.2015.6338.
- [10] P. Pokora & T. Szemberg & X. Roulleau: Bounded negativity, Harbourne constants and transversal arrangements of curves. **arXiv:1602.02379**
- [11] X. Roulleau, Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations. **arXiv:1411.6996**.
- [12] I. N. Shnurnikov, A  $t_k$  Inequality for Arrangements of Pseudolines. To appear in *Discrete Comput. Geom.*, doi:10.1007/s00454-015-9744-4
- [13] G. Urzúa, Arrangements of rational sections over curves and the varieties they define. *Rend. Lincei. Mat. Appl.* **22**: 453 – 486 (2011).

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